Failures of Intuition: Building a Solid Poker Foundation through Combinatorics

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To evaluate poker situations, the mathematics that underpin the dynamics and decisions can be incorporated into our responses experientially and intuitively. This is a common approach for most recreational players and a large fraction of professionals. It is analogous to reacting to a mathematically evolving universe through responses hewn through life experiences and evolutionarily shaped responses. This frequently works well but leads to especially poor probability estimates in cases where our experience is lacking. People make poor estimates of the frequency of rare or unprecedented events. For example, in nature we assume the progression of time is constant and have no intuition for the quantum mechanics of the microscopic universe.

In poker, these human failings manifest in many forms. One interesting way is people reacting to frequencies of actions. If a player is winning, they are typically given a lot of credit for having strong hands subsequently in that session. Conversely, the player pool reacts very differently to the same player and strategy that may lose with identical holdings and situations on alternative card run outs, expecting they can beat them as they are losing or unlucky. People misuse an intrinsic Bayesian updating; they revise their intuitive strategy without the rational interpretation required to sort fortune from folly and fundamentals. By and large, this makes it easier to win when wining and lose when losing, especially in live poker with a small sample size of hands.

Another way is in unusual card distribution situations. During a recent 5/10 no-limit hold 'em game, a dealer recounted that he had dealt a straight flush, A♦2♦3♦4♦5♦ on the board earlier that day. He asked me how likely that was to happen; he had never seen it before. I played with the math a little bit and thought it was on the order of 1/100,000 flops that any straight flush makes it on the board by the river. This did indeed fit with my limited experience; I had not seen it in years of play. I had seen plenty of boards with four to a straight flush where people win "high hand" promotions with one card. Unprompted, a confident recreational playing lawyer insisted it was millions to one. He was quickly backed up by a retired engineer playing with math sloppily. It's perhaps a sign of our times that wealthy confident people take their intuited opinions as fact, but this is a math question, so let's see what combinatorics reveals.

Straight Flush on Board

The probability of the board constituting a straight flush is identical to the frequency of being dealt a 5 card poker hand. The mathematics of this is well known but worth learning if it is unfamiliar. It can solve many problems that arise in poker, other games, and life more generally. It frees us from the shackles of our own weak intuition. The probabilities are also readily available on the web.

To begin, let us consider how many possible combinations of five card poker hands are possible choosing from a standard fifty-two card deck. This is a staple mathematical question related to binomial coefficients from algebra. (see e.g. Combination). The requisite formula is often known as "N choose M" and we will use it a lot. It gives the number of combinations of M objects in any order from N total things. For example, given three cards, say A♥K♠Q♣, there are three unordered combinations of two that can be drawn from this set: A♥K♠, A♥Q♣ and K♠Q♣. Here N=3 and M=2 so let us see how this works to get the number of combinations, C. The formula is combinations, **C= N! / (N-M)! M!** . The ! represents the factorial where, for example, 5! = 5*4*3 *2 *1 and 0! = 1! =1 by definition. The number of combinations is C= 3! / [(3-2)!*2!] = 6 / (1*2) = 3 as expected.

To construct a five card poker hand, N=52 cards and M=5 cards required to make a hand.

C(5 card poker hands)= 52! / (47!*5!) = 52*51*50*49*48/5! = 2,598,960.

This is the number of possible five card starting hands we can be dealt randomly. If we now calculate how many of them are straight flushes, the ratio will directly provide the frequency of being dealt a straight flush. It is that simple. There are ten straight flush combinations with four suits each = forty total combinations. One representative is A♦2♦3♦4♦5♦ and the lowest card can be any rank up to T♦ in T♦J♦Q♦K♦A♦.

Therefore, the frequency of a five card straight flush on the board is F:

F (straight flushes) = C(straight flushes) / C(total 5 card poker hands) = 4*10/ 2,598,960= 1/64974= 0.00001539077.

This means a straight flush will be dealt one time in (1/F) hands or on average every 1/F = 64,974 hands. A typical dealer will see on the order of 1,000 hands a week and 50,000 hands a year, the frequency is consistent with that experience.

For a harder but perhaps more intuitive way to develop this result consider a particular straight flush, A♦2 ♦3♦4♦5♦. The frequency of being dealt this exact string in this order is:

F (exact straight flush) = (1/52)*(1/51)*(1/50)*(1/49)*(1/48) = 3.20641077e-9

So this would happen only once in over 300 million hands. There are not even nearly that many five card hands, but we are insisting on getting the cards in an exact order with a particular suit. Let's relax the conditions to generalize the result. If any suit is allowed, the first card could be any ace, A♦A♥A♠A♣, this will increase the frequency by 4. But we also need to recall that they can come in any order so that gives us 5! = 120 different ways to get the same particular combination, five choices for the first card, four for the second and so on. Finally there are ten kinds of straight flushes starting with rank: A,2,3,4,5,6,7,8,9,T. Thus, the frequency for any straight is higher by a factor of (4 * 5! * 10):

F (any straight flush) = 4 * 5! * 10 * {(1/52)*(1/51)*(1/50)*(1/49)*(1/48)}= 1/64974

Exactly the same result as must be so.

Exotic Flop Frequencies

This exercise got me thinking about exotic flops and how important it might be to have a plan for them. Let's consider the relative and absolute frequency of occurrence of straight three-flush and trip flops, like 5♣ 6♣ 7♣ and A♥A♠A♣. Both are rare but familiar to most poker players.

First, calculate the total possible number of flop three card combinations using our N=52 choose M=3 formula:

C(3 card flops) = 52! / (49! 3!) = 52*51*50/3! = 22100

Straight three-flushes are 12 combinations with 4 suits each = 48 total combinations. The frequency in a three card flop is:

F (straight 3 flushes) = 4 *12 / C(3 card flops) = 12/5525= 0.0021719457

That means it will happen one time in (1/F) hands or 1/F = 460 flops on average.

The alternative way to develop this math is to consider a particular straight three-flush, A♦2 ♦3♦. The frequency of being dealt this is in this order is:

F (exact straight 3 flush) = (1/52)*(1/51)*(1/50) = 0.00000754147

Now if we want to allow it to be any suit there are 4 starting with the: A♦,A♥,A♠,A♣, this will increase the frequency by 4. But we also need to recall that they can actually come in any order so that gives us 3! = 6 different ways to get the same particular combination. Finally there are 12 strings of straight three-flushes starting with the rank A,2,3,4,5,6,7,8,9,T,J,Q.

Thus, calculate thus, the straight 3 flush flop frequency is:

F (straight 3 flushes) = 4 * 3! * 12 * F (exact straight 3 flush) = 12/5525

That's the same result as directly counting combinations.

Next, let's compare this with the probability of flopping trips on the flop. The number of combinations remains C(3 card flops) = 22,100. Any three cards of the same rank, like A♦A♥A♠, without respect to order of the flop is a triplet combination. There are 13 ranks of combinations with 4 arrangements of each set, given the four suits, for a total 52 total combinations of possible trip flops in any order. The triplet combinations are themselves calculated as N=4, M=3, C(triplets)=4! / (3! *1!) = 4 combinations or each rank triplet: A♦A♥A♠ , A♣A♥A♠ , A♦A♣A♠ , A♦A♥ A♣. The frequency in a 3 card flop is:

F (triplets)= 4 *13 / C(3 card flops) = 1/425

Meaning trips will flop one time in (1/F) hands or once every 425 hands on average.

Another way to develop this math is to consider a particular trips flop like A♦A♣ A♠. The frequency of being dealt this is in this order is:

F (exactly A♦A♣A♠) = (1/52)*(1/51)*(1/50) = 0.00000754147

Now allowing it to be any three aces this will increase the frequency by 4. But we also need to recall that they can actually come in any order so that gives us 3! = 6 different ways to get the same particular combination. Finally there are 13 strings of triplets with a distinct rank, A,2,3,4,5,6,7,8,9,T,J,Q,K. Thus, the straight three-flush flop frequency is higher:

F (triplets)= 4*3! 13 * F (exactly A♦A♣A♠) = 1/425

Thus we see it is **F (triplets)/ F (straight 3 flushes)** = 52/48 = 1.08 times as likely to see trips flop than a straight flush triplet. To get the relative frequencies it is only required to know the combinations of each. Note, both are likely to happen to a regular live player on a weekly but not daily basis.

High Hand Frequencies in Omaha and NLH

As a last combinatorics example, I'll give some insight into both high hand promotions and typical casino management. The casino I frequent once had a high hand promotion for a few hundred dollars that went to someone making quads or better. It was paid from jackpot funds raked from each pot. The ridiculous thing about it is they let Omaha poker variants participate and only required that they have a pair in their hand for a minimum qualifying holding while all quads counted for no-limit hold 'em. It doesn't take a lot of combinatoric insight (like that possessed at this point by my fine readers that made it this far) to imagine that Omaha, where one gets dealt four cards, would have a huge advantage in making strong hands. Not to mention that no-limit hold 'em hands more frequently are over on the flop and Omaha goes to the river far more often than a no-limit hold 'em hand sees the turn. As a result, the casino flooded with people playing very low stakes limit Omaha 8 Hi/Low, extracting the regular player pool prize money from the no-limit hold 'em games that were the majority contributors to the jackpot funds.

Let's estimate how unfair this decision was. First, we will estimate how frequently one makes quads if all hands go to showdown in each game. In no-limit hold 'em, one proceeds as earlier. There are 13 possible quad strings of different rank, 2-A, like A♦A♥A♠A♣X, with only one combination of each as every suit is represented. Now by showdown in no-limit hold 'em we have 7 cards to make the best 5 card hand.

C(seven card hands)= 52! / (45! 7!) = 52*51*50*49*48/5! = 133,784,560

That is a lot of combinations compared to earlier, reflecting that factorials grow exponentially. To count the quad combinations, consider a seven card sequence, (A♦A♥A♠A♣, X, X, X), where X is any of the remaining 48 cards. There are C(seven cards hands with N= 48, M=3) = 48!/(45!3!) combinations of the remaining cards that would complete the full board, and recall order doesn't matter in this counting. This gives exactly that many combinations of quad aces in seven card hands without respect to order. Now there are 13 different rank quadruplets for a total of 13*C total combinations. So the frequency is simply:

F (any quads NLH) = C(quads) / C(total hands) = 13* 48!/(45! *3!) / [52! / (45! * 7!)] = 1/ 595

So in no-limit hold 'em hands that all go to showdown, one in 595 will be quads of some variety, including any number in the hand or on the board. Note, only a relatively small number of combinations include no card from our hand as the quadruplet must be on the board, leaving only one free card combination out of 5 possible under these constraints, for 13*46=598 combinations of the 224,848. This is relevant because the high hand promotion required at least one card in your hand by one of the quads. But this contribution is negligible.

Using analogous logic to count the number of combinations of straight flushes, the number of Ace high straight flushes:

C(A high straight flushes in 7 cards) = 4*47!/(45!*2!)

Next, the higher card for the nine lower straight flushes can't be present, reducing their combinations:

C(5-K high straight flushes in 7 cards) = 4*46! / (44!* 2!)

Finally, the frequency is the ratio of the combinations of straight flush containing seven card sequences to the total number of possible seven card hands:

F (any straight flush NLH)= {1* C(A high straight flushes in 7 cards) + 9* C(2-K high straight flushes in 7 cards) } / C(7 cards hands) = [4* 47!/(45!*2!) + 9*4* 46! / (44!* 2!)] / [52! / (45! 7!)] = 113 / 363545 = 0.00031

So this happens only about once every 3,217 hands that go to showdown. Note this includes the 40 combinations of straight flushes that are themselves the board, and do not qualify for high hand. But this contribution is tiny.

Now in Omaha it is a little trickier in that we can only use two cards from our four card hand. To qualify for the high hand in Omaha, you needed a pocket pair in your hand, making the calculation a little easier. There are **C(4 card hands with N= 52, M=4) = 52!/(48!*4!) = 270,725** combinations of starting hands possible. To make quads in this fashion we must have exactly a pocket pair (or two) in our hand, the frequency of this is:

F (starting hands with pairs in Omaha)= 13*6*48! / (46 ! *2!) / C(4 card hands)= 6,768/20,825 = 0.324994

There are 13 possible pairs with six combinations each. The remaining combinatorics reflects the random nature of the other cards in a hand like AAXY, where X and Y can be any card other than an ace, leaving 46 cards from which to choose. Hands with trips in them are excluded as they cannot make quads in Omaha.

Now we have to match the starting hand of the form AAXY, with a five card board that contains AAYZT, where Y,Z,T can not be an Ace. There is exactly one combination left of our pair in the deck and it must be present on the board. For example, if we have A♦A♥XY, the remaining two Aces must appear by the river.

F (boards with our pair on them in Omaha) = 1* [46! / (43 ! *3!)] / [48! / (43!*5!)]

Here we have fixed our 1 combination of, for example, A♠A♣, and chose the other 3 cards randomly from the remaining unspecified 46 cards. We then divide by the total number of combinations of 5 card boards given the remaining 48 cards not in our original hand.

The final frequency is the product of the two frequencies:

F (quads in Omaha with a pair in our hand)= F (starting hands with pairs in Omaha)* F (boards with our pair on them in Omaha)= 12/4,165 = 0.00288

So we make quads approximately 0.3% of the time or 1/347 times.

I will spare you the math for the straight flush and refer you to the excellent site the Wizard of Odds . The straight flush, frequency = 0.000795 or approximately 1/1,258. Here, there are no issues because we always need to play two in our hand as part of a 5 card straight flush.

Thus, the a priori odds are significantly higher of making at least quads in Omaha (1/347) vs. no-limit hold ‘em (1/595) even before we consider the huge advantage in seeing 4-5 board cards far more often in Omaha. Indeed, this is an even more important factor given how combinations increase with the number of cards. Also, when the room is busy a straight flush will usually win the high hand and the relative probabilities are worse with no-limit hold ‘em (1/3217) and Omaha (1/1258).

Incidentally, the room now requires that Omaha flop the high hand making the situation far fairer. Considering quads, that doesn't change our starting hand requirements and frequency:

F (starting hands with pairs in Omaha)= 13*6*48! / (46 ! *2!) / C(4 card hands)= 6,768/20,825 = 0.324994

But now the flop requirements are more stringent with only room for 1 combination of e.g. AA and chose one other card randomly from the remaining unspecified 46 cards. We then divide by the total number of five card boards given the remaining 48 cards not in our original hand:

F (flops with our pair on them in Omaha) = 1* [46] / [48! / (45!*3!)]

The final frequency for quads on the flop in Omaha with a requisite pair in our hand, is the product:

F (quads in Omaha with a pair in our hand)= F (starting hands with pairs in Omaha)* F (flop with our pair on them in Omaha)= 18/20825 = 0.0008643, or about 1/1157 times.

So we make quads on the flop in Omaha far less often making the games competitive for high hands.

More important than the actual numbers is getting a sense of the frequencies of these rare events and knowing how to calculate the combinatorics in these common mathematically resolvable situation that arise in poker and life. Don't guess; calculate. Human were not built to understand the frequencies of rare events and make enormous errors in estimating their occurrence based on an intuition shaped by insufficient information. If you do not have time to calculate on short notice, Google the answer – a lot of folks like to calculate things and put them on the web.

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