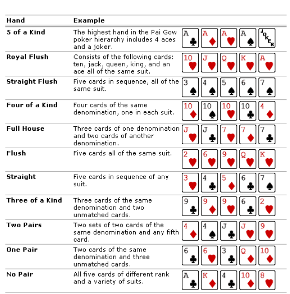
Short Deck Hand Probabilities—And the Winner is ?

Poker players, including many professionals, will grudgingly acknowledgement that poker has a mathematical foundation. Nonetheless, many professionals and most amateurs exploit the adaptivity of their brain's neural networks and respond intuitively, informed by some set of heuristics. Indeed, even top pros, playing many tables, must rely on highly informed intuition. However, it seems some mathematical assessment may be necessary as new games get introduced – at minimum the poker rules should reflect the chance of making a given hand strength in determining a showdown winner, perhaps balanced by simplicity and tradition.

Such a new game is short-deck NLH (SDNLH) that is a no-limit hold 'em, NLH, variant where the 2-5 ranked cards are removed from the deck (see e.g. https://en.wikipedia.org/wiki/Six-plus_hold_%27em). In SDNLH the game proceeds the same as NLH and the A functions as both the highest ranked card as well as a low card or "5" when making a "wheel" A6789, analogous to A2345 in full deck NLH. The evolution of the game play in SDNLH makes clear the need to revisit the simplest of mathematical foundations for poker: What hand should win at showdown and why?

Here I will show that the SDNLH probabilities are close to those in traditional pot limit Omaha (PLO), with the exception that flushes are significantly harder to make than full houses in SDNLH. Compare the results in Figures 4&5 below to see the striking similarity in probability of making a particular hand strength. In both games, three of a kind is harder to make than straights with similar absolute and relative frequencies. It is generally not recognized that the hand rankings in PLO already do not follow the difficulty of making a particular hand class (see Figure 5 below). *Given that PLO functions well under this set of rules, it is reasonable to only make the single change of prioritizing flushes over full houses and leaving straights beating three of a kind.* In short deck PLO (see Figure 6 below) the situation is a bit different as straights are much easier to make than three of a kind, yet changing the rankings between the variants using the same deck seems unappealing.

Consider, SDNLH was initially played with the same hand rankings as NLH, where a straight flush is the best hand and high card the weakest possible winning hand, with the traditional rankings in between. It turns out that in a 7 card poker game like SDNLH, flushes are significantly harder to make than full houses, with fewer suited cards remaining in the smaller deck. Further,



Hand	Example	
5 of a Kind	The highest hand in the Pai Gow poker hierarchy includes 4 aces and a joker.	A A A A JOKER
Royal Flush	Consists of the following cards: ten, jack, queen, king, and an ace all of the same suit.	10 J Q K A
Straight Flush	Five cards in sequence, all of the same suit.	3 4 5 6 7
Four of a Kind	Four cards of the same denomination, one in each suit.	10 10 10 10 4
Full House	Three cards of one denomination and two cards of another denomination.	J J 7 7 7
Flush	Five cards all of the same suit.	2 6 9 Q K
Straight	Five cards in sequence of any suit.	3 4 5 6 7
Three of a Kind	Three cards of the same denomination and two unmatched cards.	9 9 9 6 2
Two Pairs	Two sets of two cards of the same denomination and any fifth card.	4 4 J J 9
One Pair	Two cards of the same denomination and three unmatched cards.	6 6 3 Q 10
No Pair	All five cards of different rank and a variety of suits.	A K 4 10 8

Figure 1: Traditional poker hand rankings evolved from experience with 5 card poker games.

straights are more common than three-of-a-kind in the 7 card games, but harder to make than trips in a 5 card game. Thus, people were playing the game in an unusual way. Imagine if NLH were played such that two pair beat a flush, and how that would change the incentives. Hand rankings reflecting the difficulty of making a particular holding is a critical foundation or at least a guiding principle of poker games. Nonetheless, consider that in both full deck and short deck poker hand rankings do not remain consistent throughout different common variants when following relative probabilities of making a particular hand, it matters what specific 5 or 7 card game one is playing to decide the winning hand type if it were based on frequency of occurrence!

Let's take look at the combinatorics that determine hand strengths. The rankings in between a royal flush and high card in a five card poker hand are determined by the difficulty in making a particular poker hand. In full deck poker, the hand rankings, pair and above, remain fixed over the different 5 and 7-card variants that are typically spread. Figure 1 (https://en.wikipedia.org/wiki/List_of_poker_hands) lists the hand rankings and gives an example of each.

Implied in this ranking is that the relative difficulty in making each hand is preserved in each poker variant. However, 5-card and 7-card stud probabilities for making each particular hand will be different; similar considerations will result in NLH and pot-limit Omaha (PLO) hands having different frequencies of occurrence. Let's consider five and seven card stud for clarity. With five cards there are N= 52! / (47! x 5!) =2,598,960 combinations of hands and given 7 cards there are N= 52! / (45! x 7!) = 133,784,560. Let's see how hard it is to make a royal flush for each. Considering five card hands there are only 4 combinations, so the probability of making one is P= 4 / 2,598,960, or one time in 649,740 hands. From a seven card perspective, there are still the same 4 combinations of 5 cards that need to be amongst the seven cards, but now there two arbitrary cards giving C=47! / (45! 2!) combinations of side cards and thus C=4324 possible royal flush combinations. This leads to a probability P= 4324/ 133,784,560, or one in 30,940 hands will be a royal flush. The list of exact frequencies and probabilities for five and seven card poker hands is reproduced here in Figure 2&3 from https://en.wikipedia.org/wiki/Poker_probability.

Hand	Distinct hands	Frequency	Probability	Cumulative probability	Odds	Mathematical expression of absolute frequency
Royal flush	1	4	0.000154%	0.000154%	649,740 : 1	$\binom{4}{1}$
Straight flush (excluding royal flush)	9	36	0.00139%	0.0015%	72,192 : 1	$\binom{10}{1}\binom{4}{1}-\binom{4}{1}$
Four of a kind	156	624	0.0240%	0.0256%	4,165 : 1	$\binom{13}{1}\binom{12}{1}\binom{4}{1}$
Full house	156	3,744	0.1441%	0.17%	693 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$
Flush (excluding royal flush and straight flush)	1,277	5,108	0.1965%	0.367%	508 : 1	$\binom{13}{5}\binom{4}{1}-\binom{10}{1}\binom{4}{1}$
Straight (excluding royal flush and straight flush)	10	10,200	0.3925%	0.76%	254 : 1	$\binom{10}{1}\binom{4}{1}^5-\binom{10}{1}\binom{4}{1}$
Three of a kind	858	54,912	2.1128%	2.87%	46.3 : 1	$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2$
Two pair	858	123,552	4.7539%	7.62%	20.0 : 1	$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}$
One pair	2,860	1,098,240	42.2569%	49.9%	1.37 : 1	$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3$
No pair / High card	1,277	1,302,540	50.1177%	100%	0.995 : 1	$\left[\binom{13}{5}-10\right]\left[\binom{4}{1}^5-4\right]$
Total	**7,462**	**2,598,960**	**100%**	---	**0 : 1**	$\binom{52}{5}$

Figure 2: Hand rankings and probabilities for 5 card poker games like five card stud.

The table of five card hands in Figure 2 shows that the traditional hand rankings reflect the relative probabilities of making a five card hand. It is most common to end up with no pair given five random cards, and this happens about half the time. Further, it gets progressively more difficult to make higher ranked hands. While we make a pair a little less than half the time, two pair is a rare, 20:1, occurrence, while flushes and full houses are hundreds to one against. To make quads, forget about it, at over 4000:1 and straight flushes are over ten times more difficult. One would guess that these stark differences in frequency led to the intuitive development of hand rankings in poker.

However, if we now look at the frequencies, from exact combinatorics, for seven card hands in Figure 3, things change substantially. Interestingly, with seven card hands it is easier to make a pair than no pair, so we get

Hand	Frequency	Probability	Cumulative	Odds
Royal flush	4,324	0.0032%	0.0032%	30,939 : 1
Straight flush (excl. royal flush)	37,260	0.0279%	0.0311%	3,589.6 : 1
Four of a kind	224,848	0.168%	0.199%	594 : 1
Full house	3,473,184	2.60%	2.80%	37.5 : 1
Flush	4,047,644	3.03%	5.82%	32.1 : 1
Straight	6,180,020	4.62%	10.4%	20.6 : 1
Three of a kind	6,461,620	4.83%	15.3%	19.7 : 1
Two pair	31,433,400	23.5%	38.8%	3.26 : 1
One pair	58,627,800	43.8%	82.6%	1.28 : 1
No pair	23,294,460	17.4%	100%	4.74 : 1
Total	133,784,560	100%	100%	0 : 1

Figure: 3 Hand rankings and probabilities for 7 card poker games like seven card stud or NLH.

an inversion compared to 5 card hands. Indeed, it is slightly easier to end up with two pair rather than no pair in a seven card hand. Still, the rankings of higher ranked, aspirational hands have relative probabilities that reflect the same hierarchy. However, in the seven card hands the relative probabilities are no longer well separated and strong hands are fairly common. For example, three of a kind and straights have similar frequencies and occur with an absolute frequency of roughly 5%; flushes and full house are more than half as likely. The exotic holdings like quads are now something one expects to make on a monthly basis in live poker play. Based on the math, one could imagine valuing no pair over paired holdings and having a meaningful game mathematically. Yet, the game flow dynamics of poker are still consistent and reasonable with these inverted probabilities – one discards weaker hands earlier in game play trying to make stronger hands to win a pot while hand ranges strengthen through evolutionary selection. While it is harder to make no pair than a single pair it cannot develop into a stronger hand absent other draws. The frequencies for better holdings above no pair, i.e. high card, are consistent with the hand rankings. It is worth noting how distinctly the math changes with the seven card combinatorics relative to five card hands and how that informs strategic play.

Next, let's consider the new SDNLH and the associated combinatorics. The exact mathematics was summarized by contributor whosnext on the twoplustwo forums and the results are reproduced in Figure 4. (https://forumserver.twoplustwo.com/25/probability/six-plus-hold-em-hand-rankings-1685367/) If one stays with the assumption that the relative frequency and associated probability of making each hand determines the rankings of winning hands, flushes will always beat full houses. Interestingly, from five to seven card hands, three of a kind becomes roughly twice as hard to make relative to straights, a flip from the opposite similar relative frequency for five card hands. Thus, at first glance it seems to make little sense to value straights over three of a kind as a standard for a SDNLH game.

Six Plus Category	Five Card Freq	Five Card Pct	Seven Card Freq	Seven Card Pct
Royal Flush	4	0.00	1,860	0.02
Straight Flush	20	0.01	8,700	0.10
Four of a Kind	288	0.08	44,640	0.53
Flush	480	0.13	175,560	2.10
Full House	1,728	0.46	633,024	7.58
Three of a Kind	16,128	4.28	637,560	7.64
Straight	6,120	1.62	1,139,580	13.65
Two Pair	36,288	9.63	3,157,056	37.82
One Pair	193,536	51.34	2,316,600	27.75
High Card	122,400	32.47	233,100	2.79
TOTAL	376,992	100.00	8,347,680	100.00

Figure 4: Hand rankings and probabilities for 5 and 7 short deck card poker games. SDNLH, 6 Plus Hold 'Em, is a 7 card game. 5 card games are not currently widely spread.

However, if we consider normal deck PLO in Figure 5, the same reversal of frequency and winning hand strength is present with a similar inversion in probability albeit it is about twice as pronounced in SDNLH. Nonetheless, it seems to be sensible to preserve the evolutionary structure of poker hands -- a person flopping trips and competing to beat straights by trying to make a full house. The statistical inversion does not make standard PLO a pathologically flawed game but rather it is implicitly a part of the allure of the great game of PLO.

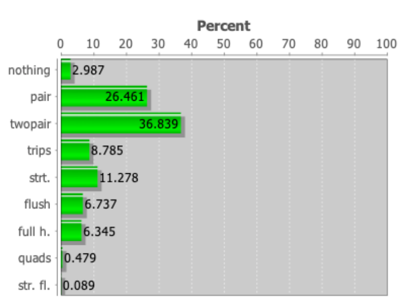


Figure 5: Traditional PLO hand probabilities simulated.

Recently, people also started playing short deck pot limit Omaha, SDPLO, which would again alter the combinatorics as one has to account for the condition of only using two of four cards in a hand in combination with the five board cards. I gave examples of the mathematics involved in similar calculations in a previous article (https://www.twoplustwo.com/magazine/issue164/brian-spacefailures-of-intuition.php). I'll leave the exact calculations as a future exercise and others have outlined the math in detail like presented here: http://people.math.sfu.ca/~alspach/computations.html.

But to get a feeling for the game, one can approximately calculate the probabilities using ProPokerTools and do the math numerically exactly except that it will miss a small set of wheel straights, A6789. This can be done in SDNLH by setting a single player card range to **([6+][6+]) with the Dead cards:5s4s3s2s5h4h3h2h5d4d3d2d5c4c3c2c**. In SDPLO the single player range is set to: **([6+][6+][6+],[6+])** . After averaging over 300 million five card boards (for both the results in Figures 5&6) and assessing the winning hand type the probabilities are produced. Note, SDNLH was used as a control simulation and missing the wheel straights had a minor effect of the hand probabilities below straights, compared to the exact results in Figure 4; wheels are miscounted as high card or pair hands.

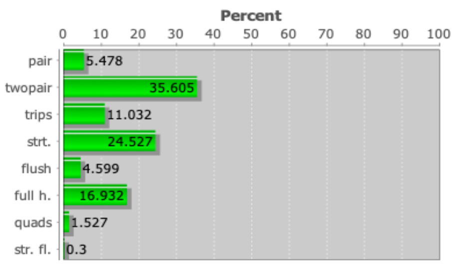


Figure 6: Short deck PLO probabilities with wheels not counted.

Thus, looking at the math of SDNLH, it seems reasonable and convenient to only change the hand rankings of flushes relative to the other holdings. Figure 7 summarizes the situation comparing the hand frequencies – SDNLH and standard deck PLO are very similar aside from the lower flush frequency in SDNLH. While three of a kind is harder to make than straights in SDNLH, this is already the case in standard PLO. Keeping straights as beating trips preserves the nature of hand evolution in the poker game. For example, if one player flops a set and the other a straight or straight draw, the poker dynamics of these two similar frequency hands interacting is an important poker feature; three of a kind can reasonably hope to turn into a full house and win. This creates a foundational poker structure with a full house beating a straight itself winning vs three of a kind while these hands put pressure on lesser holdings. ***So in sum, in the spirit of minimal change and do no harm, simply promote the flush over the full house, where a new tension is created, preserving poker dynamics with only a minor change in structure for all the new short deck games including SDNLH and SDPLO; leave straights beating three of a kind.***

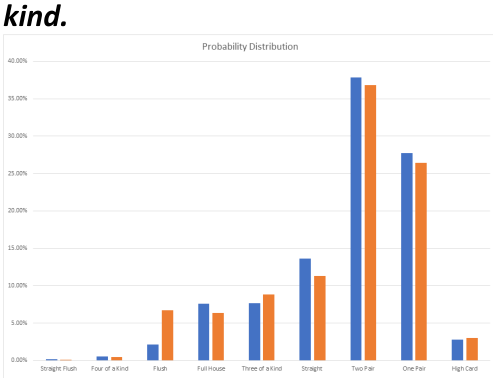


Figure 7 Frequencies of Hands ***SDNLH*** *vs regular* ***PLO***

As an aside, I have an interest in poker theory. I believe that the nature of the optimal solutions is not yet completely understood. It seems to me from an information theoretic perspective that bet sizing should draw from distributions, providing information hiding. Solver work suggests this might be right. I also see an analogy between (statistical) mechanical energy and expected value in that the solution space of poker is a surface of constant expected value. Lastly, I have noted that the game theoretical optimal strategy is the one the requires no information of the opponents play suggesting a Shannon entropy tie in. All of *this suggests to me a statistical mechanical approach to poker solutions that I have not formulated. If anyone is interested pursuing this feel free to get in touch.*

Brian Space is a scientist and professor seeking people to play Quantum Statistical Mechanics for money. He plays poker in the Tampa Bay Florida area. His poker articles are available on his web site: http://drbrian.space/poker.htm